Χρήσιμες Ταυτότητες

Αναπτύγματα

- $(x+y)^2 = x^2 + 2xy + y^2$
- $(x-y)^2 = x^2 - 2xy + y^2$
- $(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$
- $(x-y)^3 = x^3 - 3x^2y + 3xy^2 - y^3$

Αθροίσματα και Διαφορές

- $x^2 - y^2 = (x-y)(x+y)$
- $x^3 - y^3 = (x-y)(x^2+xy+y^2)$
- $x^3 + y^3 = (x+y)(x^2-xy+y^2)$

Διωνυμικό Ανάπτυγμα

Τύπος Διωνύμου

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$

Διωνυμικοί Συντελεστές

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} = \binom{n}{n-k}$$

Ιδιότητες Διωνυμικών Συντελεστών

- $\binom{n}{n} = 1$
- $\binom{n}{1} = \binom{n}{n-1} = n$
- $\binom{n}{k} + \binom{n}{k+1} = \binom{n+1}{k+1}$

Τρίγωνο του Pascal

$$(a+b)^0 = 1$$
$$(a+b)^1 = 1a + b1$$
$$(a+b)^2 = 1a^2 + 2ab + b^21$$
$$(a+b)^3 = 1a^3 + 3a^2b + 3ab^2 + b^31$$
$$(a+b)^4 = 1a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^41$$

Όρια Συναρτήσεων

Ιδιότητες των Ορίων

- $\lim\limits_{x\to x_0} c = c$
- $\lim\limits_{x\to x_0} x = x_0$
- $\lim\limits_{x\to x_0} (f(x) \pm g(x)) = \lim\limits_{x\to x_0} f(x) \pm \lim\limits_{x\to x_0} g(x)$
- $\lim\limits_{x\to x_0} (a \cdot f(x)) = a \cdot \lim\limits_{x\to x_0} f(x)$
- $\lim\limits_{x\to x_0} (f(x) \cdot g(x)) = \lim\limits_{x\to x_0} f(x) \cdot \lim\limits_{x\to x_0} g(x)$
- $\lim\limits_{x\to x_0} \left(\frac{f(x)}{g(x)}\right) = \frac{\lim\limits_{x\to x_0} f(x)}{\lim\limits_{x\to x_0} g(x)}, \quad \lim\limits_{x\to x_0} g(x) \neq 0$
- $\lim\limits_{x\to x_0} (f(x))^n = \left(\lim\limits_{x\to x_0} f(x)\right)^n, \quad n \in \mathbb{N}$
- $\lim\limits_{x\to x_0} \sqrt[n]{f(x)} = \sqrt[n]{\lim\limits_{x\to x_0} f(x)}, \quad n \in \mathbb{N}^*$

Χρήσιμα Όρια

- $\lim\limits_{x\to\infty} \sqrt[x]{a} = 1,\ a > 0$
- $\lim\limits_{x\to\infty} \sqrt[x]{x} = 1$
- $\lim\limits_{x\to 0} \frac{\sin x}{x} = 1$
- $\lim\limits_{x\to\infty} \frac{\sin x}{x} = 0$
- $\lim\limits_{x\to\infty} \left(1 + \frac{1}{x}\right)^x = e$
- $\lim\limits_{x\to\infty} \left(1 + \frac{a}{x}\right)^x = e^a$

Κανόνας L' Hôpital

$$\lim_{x\to x_0} \frac{f(x)}{g(x)} \overset{\left(\frac{0}{0}\right)}{\underset{\left(\frac{\infty}{\infty}\right)}{=}} \lim_{x\to x_0} \frac{f'(x)}{g'(x)}$$

Συνέχεια Συνάρτησης

f συνεχής στο $x_0 \Leftrightarrow \lim\limits_{x\to x_0} f(x) = f(x_0)$

Κριτήριο Παρεμβολής

$$\left.\begin{array}{l} \text{(i)}\ \ f(x) \le g(x) \le h(x) \\ \text{(ii)}\ \ \lim\limits_{x\to x_0} f(x) = \lim\limits_{x\to x_0} h(x) = l \end{array}\right\} \Rightarrow \lim_{x\to x_0} g(x) = l$$

Μηδενική επί Φραγμένη

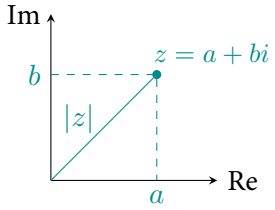
$$\left.\begin{array}{l} \text{(i)}\ \ \lim\limits_{x\to x_0} g(x) = 0 \\ \text{(ii)}\ \ f(x) \text{ φραγμένη} \end{array}\right\} \Rightarrow \lim_{x\to x_0} f(x) \cdot g(x) = 0$$

ΠΡΟΣΗΜΟ

Μιγαδικοί Αριθμοί

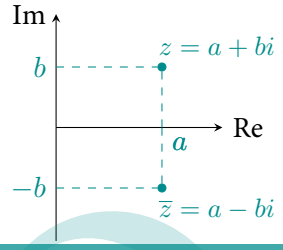
Μέτρο Μιγαδικού Αριθμού

$$|z| = \sqrt{a^2 + b^2}$$



Συζυγής Μιγαδικού Αριθμού

$$\overline{z} = a - bi$$



Ιδιότητες του Μέτρου

- $|z| \geq 0$ και $|z| = 0 \Leftrightarrow z = 0$
- $|z| = |-z| = |\overline{z}| = |-\overline{z}|$
- $||z_1| - |z_2|| \leq |z_1 \pm z_2| \leq |z_1| + |z_2|$
- $|z_1 \cdot z_2| = |z_1| \cdot |z_2|$
- $\left|\dfrac{z_1}{z_2}\right| = \dfrac{|z_1|}{|z_2|}, \; z_2 \neq 0$

Ιδιότητες του Συζυγούς

- $\overline{\overline{z}} = z$
- $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$
- $\overline{z_1 \cdot z_2} = \overline{z_1} \cdot \overline{z_2}$
- $\overline{\left(\dfrac{z_1}{z_2}\right)} = \dfrac{\overline{z_1}}{\overline{z_2}}, \; z_2 \neq 0$
- $z \cdot \overline{z} = |z|^2$
- $z = \overline{z} \Leftrightarrow z \in \mathbb{R}$
- $\operatorname{Re} z = \dfrac{z + \overline{z}}{2}$
- $\operatorname{Im} z = \dfrac{z - \overline{z}}{2i}$

Πράξεις Μιγαδικών Αριθμών

Αν $z_1 = a + bi$ και $z_2 = c + di$

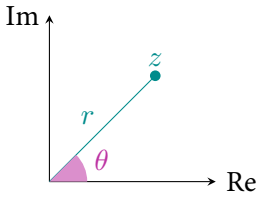
- $z_1 \pm z_2 = (a \pm c) + (b \pm d)i$
- $z_1 \cdot z_2 = (ac - bd) + (ad + bc)i$
- $\dfrac{z_1}{z_2} = \dfrac{ac + bd}{c^2 + d^2} + \dfrac{bc - ad}{c^2 + d^2} i$

Πολική μορφή

$$z = r(\cos\theta + i\sin\theta)$$

Εκθετική μορφή

$$z = re^{i\theta}$$



Θεώρημα De Moivre

$$z^n = r^n\big(\cos(n\theta) + i\sin(n\theta)\big) \quad , \quad n \in \mathbb{N}^*$$

Ρίζες Μιγαδικού Αριθμού

$$z_k = \sqrt[n]{r}\left(\cos\frac{\theta + 2k\pi}{n} + i\sin\frac{\theta + 2k\pi}{n}\right)$$

Δυνάμεις της Φανταστικής Μονάδας

- $i^0 = 1$
- $i^1 = i$
- $i^2 = -1$
- $i^3 = i^2 \cdot i = -i$
- $i^{4n} = 1$
- $i^{4n+1} = i$
- $i^{4n+2} = -1$
- $i^{4n+3} = -i$

Ιδιότητες των Δυνάμεων

- $z^m \cdot z^n = z^{m+n}$
- $\dfrac{z^m}{z^n} = z^{m-n}$
- $(z^m)^n = z^{m \cdot n}$
- $(z_1 \cdot z_2)^n = z_1^n \cdot z_2^n$

Ρίζα της Φανταστικής Μονάδας

$$\sqrt{\pm i} = \frac{\sqrt{2}}{2}(1 \pm i)$$

ΠΡΟΣΧΕΔΙΟ

Παράγωγοι

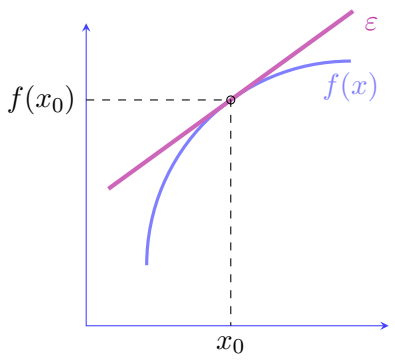
Παράγωγοι Βασικών Συναρτήσεων

- $\frac{d}{dx}(c) = 0, \quad c \in \mathbb{R}$
- $\frac{d}{dx}(x) = 1$
- $\frac{d}{dx}(x^a) = ax^{a-1}, \quad a \in \mathbb{R} \setminus \{-1\}$
- $\frac{d}{dx}\left(\frac{1}{x}\right) = -\frac{1}{x^2}$
- $\frac{d}{dx}\left(\frac{1}{x^2}\right) = -\frac{2}{x^3}$
- $\frac{d}{dx}(a^x) = a^x \cdot \ln a$
- $\frac{d}{dx}(e^x) = e^x$
- $\frac{d}{dx}(\ln x) = \frac{1}{x}$
- $\frac{d}{dx}\left(\sqrt[n]{x^m}\right) = \frac{d}{dx}\left(x^{\frac{m}{n}}\right) = \frac{m}{n}x^{\frac{m}{n}-1}$
- $\frac{d}{dx}(\sqrt{x}) = \frac{1}{2\sqrt{x}}$
- $\frac{d}{dx}(\sin x) = \cos x$
- $\frac{d}{dx}(\cos x) = -\sin x$
- $\frac{d}{dx}(\tan x) = \frac{1}{\cos^2 x}$
- $\frac{d}{dx}(\cot x) = -\frac{1}{\sin^2 x}$
- $\frac{d}{dx}(\arctan x) = \frac{1}{1+x^2}$
- $\frac{d}{dx}(\operatorname{arccot} x) = \frac{-1}{1+x^2}$
- $\frac{d}{dx}(\arcsin x) = \frac{1}{\sqrt{1-x^2}}$
- $\frac{d}{dx}(\arccos x) = \frac{-1}{\sqrt{1-x^2}}$
- $\frac{d}{dx}(\sinh x) = \cosh x$
- $\frac{d}{dx}(\cosh x) = \sinh x$

Κανόνες Παραγώγισης

- $\frac{d}{dx}(af(x)) = a\frac{d}{dx}(f(x))$
- $\frac{d}{dx}(f(x)+g(x)) = \frac{d}{dx}(f(x)) + \frac{d}{dx}(g(x))$
- $\frac{d}{dx}(f(x)\cdot g(x)) = \frac{d}{dx}(f(x))\cdot g(x) + f(x)\cdot\frac{d}{dx}(g(x))$
- $\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{\frac{d}{dx}(f(x))\cdot g(x) - f(x)\cdot\frac{d}{dx}(g(x))}{g^2(x)}$
- $\frac{d}{dx}(f(g(x))) = \frac{d}{dg(x)}(f(g(x)))\cdot\frac{d}{dx}(g(x))$

Εξίσωση Εφαπτόμενης ευθείας



$\varepsilon : y - f(x_0) = f'(x_0)(x - x_0)$

Τύπος Taylor

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + R_n(x)$$

Τύπος Υπολοίπου κατά Lagrange

$$R_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-a)^{n+1}, \quad \text{όπου } \xi \in (a,x) \text{ ή } \xi \in (x,a)$$

Τύπος Maclaurin

$$f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + R_n(x)$$

ΠΡΟΣΗΜΟ

Ολοκληρώματα

Ολοκληρώματα Βασικών Συναρτήσεων

- $\int 0\,dx = c, \quad c \in \mathbb{R}$
- $\int dx = x + c \int x^n\,dx = \frac{x^{n+1}}{n+1} + c, \quad n \in \mathbb{N}$
- $\int \frac{1}{x^n}\,dx = \int x^{-n}\,dx = \frac{x^{-n+1}}{-n+1} + c$
- $\int \frac{1}{x}\,dx = \ln|x| + c$
- $\int \frac{1}{x^2}\,dx = -\frac{1}{x} + c$
- $\int a^x\,dx = \frac{a^x}{\ln a} + c$
- $\int e^x\,dx = e^x + c$
- $\int \sqrt[n]{x^m}\,dx = \int x^{\frac{m}{n}}\,dx = \frac{x^{\frac{m}{n}+1}}{\frac{m}{n}+1} + c$
- $\int \sqrt{x}\,dx = \int x^{\frac{1}{2}}\,dx = \frac{2}{3}x^{\frac{3}{2}} + c$
- $\int \sin x\,dx = -\cos x + c$
- $\int \cos x\,dx = \sin x + c$
- $\int \frac{1}{\cos^2 x}\,dx = \tan x + c$
- $\int \frac{1}{\sin^2 x}\,dx = -\cot x + c$
- $\int \frac{1}{1+x^2}\,dx = \arctan x + c$
- $\int \frac{-1}{1+x^2}\,dx = \operatorname{arccot} x + c$
- $\int \frac{1}{\sqrt{1-x^2}}\,dx = \arcsin x + c$
- $\int \frac{-1}{\sqrt{1-x^2}}\,dx = \arccos x + c$
- $\int \sinh x\,dx = \cosh x + c$
- $\int \cosh x\,dx = \sinh x + c$

Ολοκλήρωση κατά Παράγοντες

$$\int f'(x)\cdot g(x)\,dx = f(x)\cdot g(x) - \int f(x)\cdot g'(x)\,dx$$

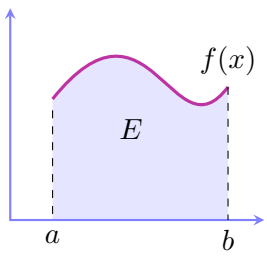
Κατά Παράγοντες (Ειδική Περίπτωση)

$$\int f'(x)\cdot f(x)\,dx = \frac{f^2(x)}{2} + c$$

Ολοκλήρωση με Αντικατάσταση

$$\int f(u)\,du = F(u) \Rightarrow \int f(g(x))\cdot g'(x)\,dx = F(g(x))$$

Ορισμένο Ολοκλήρωμα



$$E = \int_a^b f(x)\,dx = \left[F(x)\right]_a^b = F(b) - F(a)$$

Γενικευμένο Ολοκλήρωμα

I είδους

- $\int_a^{+\infty} f(x)\,dx = \left[F(x)\right]_a^{+\infty} = \lim_{x\to+\infty} F(x) - F(a)$
- $\int_{-\infty}^{b} f(x)\,dx = \left[F(x)\right]_{-\infty}^{b} = F(b) - \lim_{x\to-\infty} F(x)$
- $\int_{-\infty}^{+\infty} f(x)\,dx = \int_{-\infty}^{0} f(x)\,dx + \int_{0}^{+\infty} f(x)\,dx$

II είδους

- $\int_a^{b^-} f(x)\,dx = \left[F(x)\right]_a^{b^-} = \lim_{x\to b^-} F(x) - F(a)$
- $\int_{a^+}^{b} f(x)\,dx = \left[F(x)\right]_{a^+}^{b} = F(b) - \lim_{x\to a^+} F(x)$
- $\int_{a^+}^{b^-} f(x)\,dx = \int_{a^+}^{c} f(x)\,dx + \int_{c}^{b^-} f(x)\,dx$

ΠΡΟΣΗΜΟ

Τριγωνομετρία

Βασικές Ταυτότητες

- $\sin^2 A + \cos^2 A = 1$
- $\tan A = \dfrac{\sin A}{\cos A}$
- $\cot A = \dfrac{1}{\tan A} = \dfrac{\cos A}{\sin A}$

Μοίρες και ακτίνια

$$\boxed{\frac{a}{\pi} = \frac{\mu^\circ}{180^\circ}}$$

Τύποι Διπλάσιου Τόξου

- $\sin 2A = 2 \sin A \cos A$
- $\cos 2A = \cos^2 A - \sin^2 A$
- $\tan 2A = \dfrac{2 \tan A}{1 - \tan^2 A}$

Τύποι Αποτετραγωνισμού

- $\sin^2 A = \dfrac{1 - \cos 2A}{2}$
- $\cos^2 A = \dfrac{1 + \cos 2A}{2}$

Τριγωνομετρικοί Αριθμοί Οξειών Γωνιών

Γωνία A	Γωνία A	$\sin A$	$\cos A$	$\tan A$	$\cot A$
0°	0	0	1	0	∞
30°	$\pi/6$	$1/2$	$\sqrt{3}/2$	$\sqrt{3}/3$	$\sqrt{3}$
45°	$\pi/4$	$\sqrt{2}/2$	$\sqrt{2}/2$	1	1
60°	$\pi/3$	$\sqrt{3}/2$	$1/2$	$\sqrt{3}$	$\sqrt{3}/3$
90°	$\pi/2$	1	0	$\pm\infty$	0

Αναγωγή στο 1ο Τεταρτημόριο

	$-A$	$\pi/2 \pm A$	$\pi \pm A$	$3\pi/2 \pm A$
sin	$-\sin A$	$\cos A$	$\mp \sin A$	$-\cos A$
cos	$\cos A$	$\mp \sin A$	$-\cos A$	$\mp \sin A$
tan	$-\tan A$	$\mp \cot A$	$\pm \tan A$	$\mp \cot A$
cot	$-\cot A$	$\mp \tan A$	$\pm \cot A$	$\mp \tan A$

Τύποι Αθροίσματος - Διαφοράς

- $\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$
- $\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$
- $\tan(A \pm B) = \dfrac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$
- $\sin A \pm \sin B = 2 \sin \dfrac{A \pm B}{2} \cos \dfrac{A \mp B}{2}$
- $\cos A + \cos B = 2 \cos \dfrac{A + B}{2} \cos \dfrac{A - B}{2}$
- $\cos A - \cos B = 2 \sin \dfrac{A + B}{2} \sin \dfrac{B - A}{2}$

Τύποι Τριπλάσιου Τόξου

- $\sin 3A = 3 \sin A - 4 \sin^3 A$
- $\cos 3A = 4 \cos^3 A - 3 \cos A$
- $\tan 3A = \dfrac{3 \tan A - \tan^3 A}{1 - 3 \tan^2 A}$

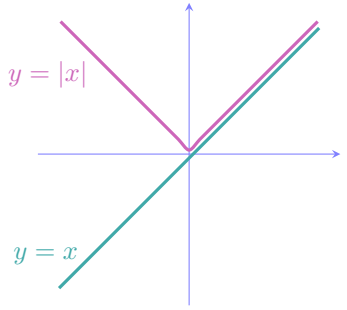
Τύποι Γινομένου

- $\sin A \sin B = \dfrac{1}{2}\{\cos(A - B) - \cos(A + B)\}$
- $\cos A \cos B = \dfrac{1}{2}\{\cos(A - B) + \cos(A + B)\}$
- $\sin A \cos B = \dfrac{1}{2}\{\sin(A - B) + \sin(A + B)\}$

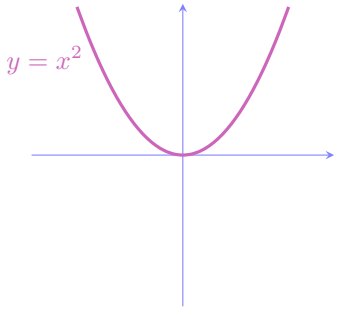
ΠΡΟΣΗΜΟ

Βασικές Συναρτήσεις

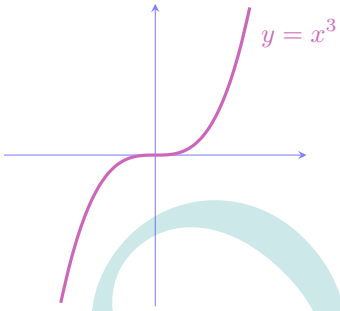
Απόλυτη τιμή



Άρτια Πολυωνυμική

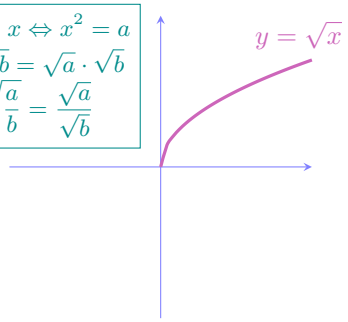


Περιττή Πολυωνυμική

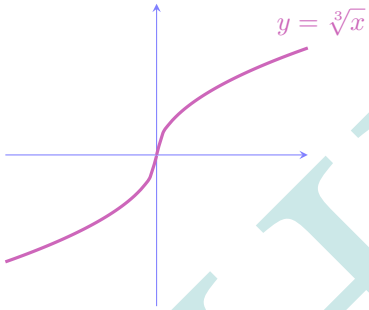


Άρτια Ρίζα

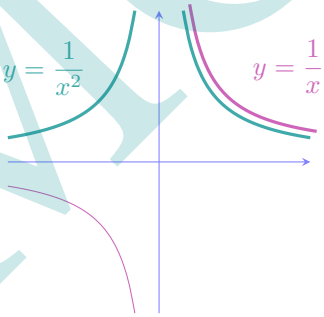
$$\sqrt{a} = x \Leftrightarrow x^2 = a$$
$$\sqrt{a \cdot b} = \sqrt{a} \cdot \sqrt{b}$$
$$\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$$



Περιττή Ρίζα

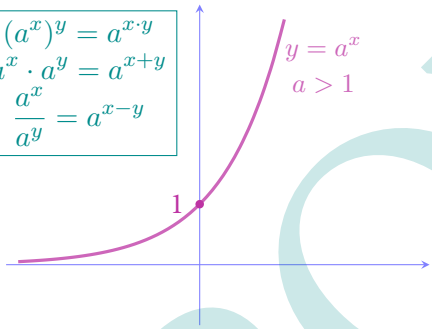


Υπερβολή

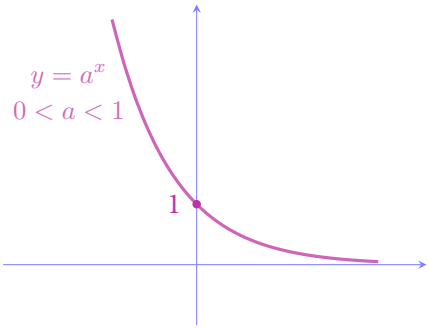


Αύξουσα Εκθετική

$$(a^x)^y = a^{x \cdot y}$$
$$a^x \cdot a^y = a^{x+y}$$
$$\frac{a^x}{a^y} = a^{x-y}$$

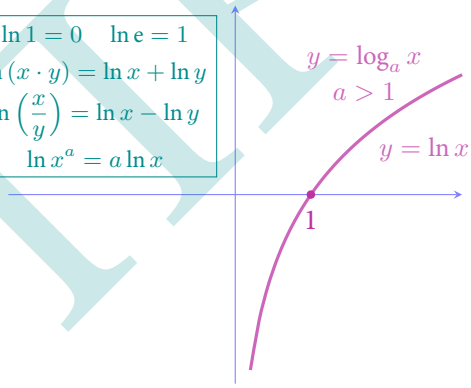


Φθίνουσα Εκθετική

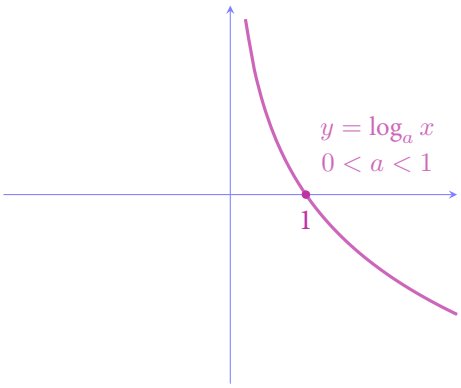


Αύξουσα Λογαριθμική

$$\ln 1 = 0 \qquad \ln \mathrm{e} = 1$$
$$\ln(x \cdot y) = \ln x + \ln y$$
$$\ln\left(\frac{x}{y}\right) = \ln x - \ln y$$
$$\ln x^a = a \ln x$$



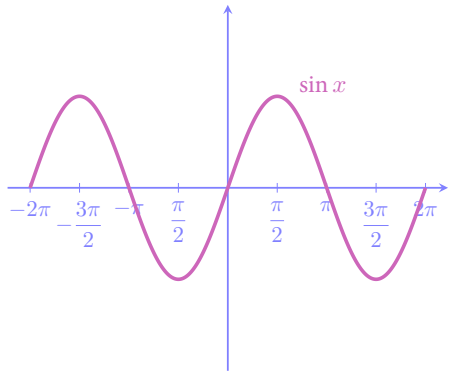
Φθίνουσα Λογαριθμική



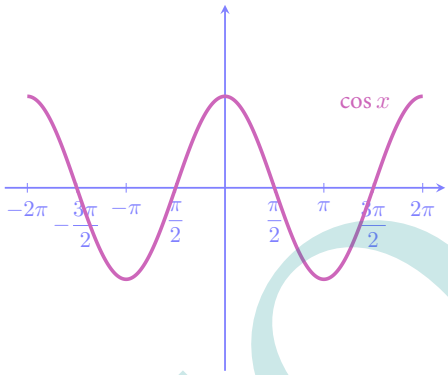
ΠΡΟΣΗΜΟ

Τριγωνομετρικές Συναρτήσεις

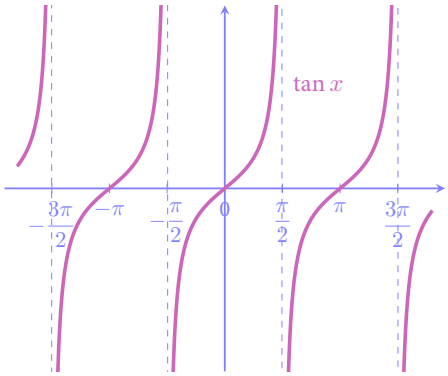
Ημίτονο



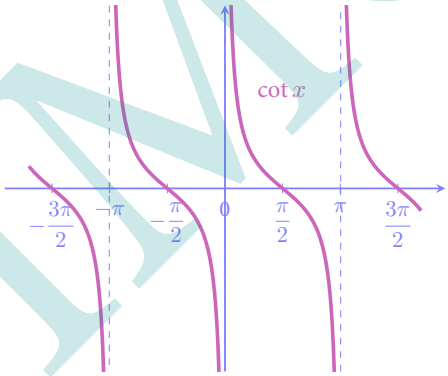
Συνημίτονο



Εφαπτομένη

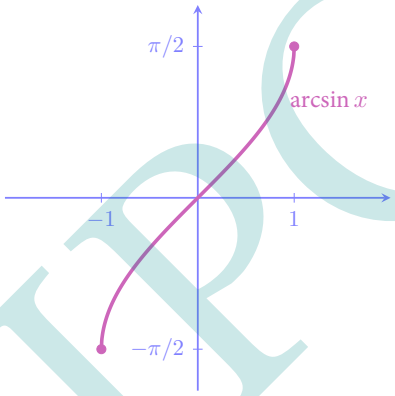


Συνεφαπτομένη



Αντίστροφες Τριγωνομετρικές Συναρτήσεις

Αντίστροφη Ημιτόνου



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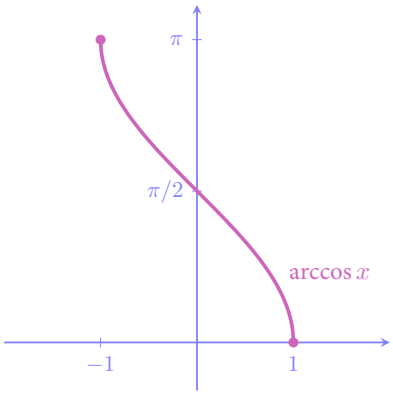
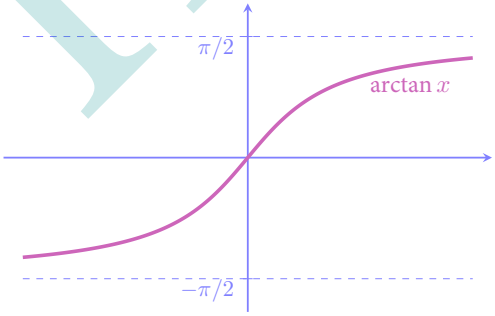
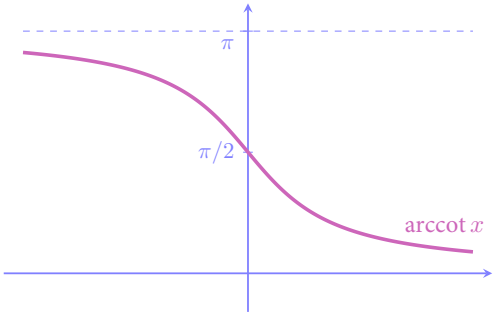


Αντίστροφη Εφαπτομένης



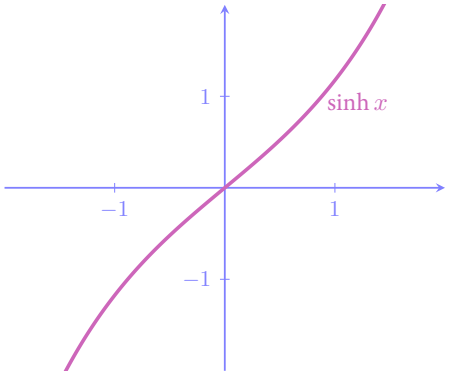
Αντίστροφη Συνεφαπτομένης



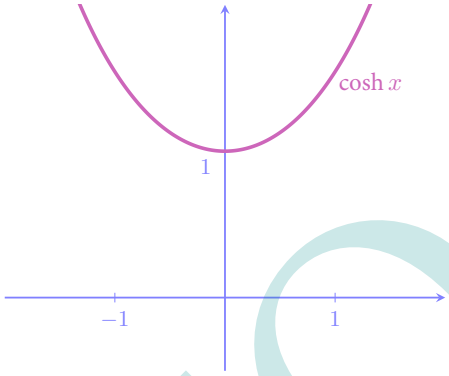
ΠΡΟΣΧΗΜΟ

Υπερβολικές Συναρτήσεις

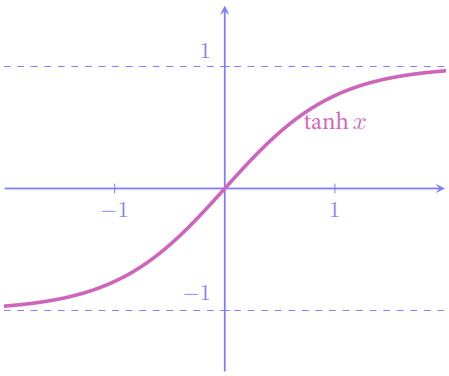
Υπερβολικό Ημίτονο



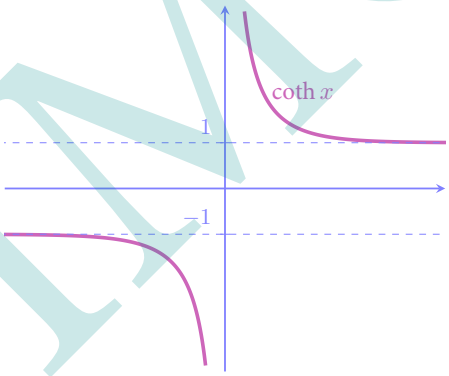
Υπερβολικό Συνημίτονο



Υπερβολική Εφαπτομένη

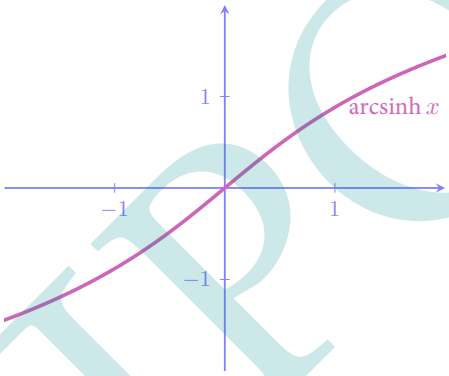


Υπερβολική Συνεφαπτομένη

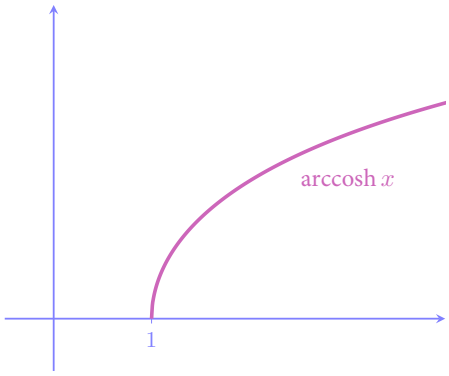


Αντίστροφες Υπερβολικές Συναρτήσεις

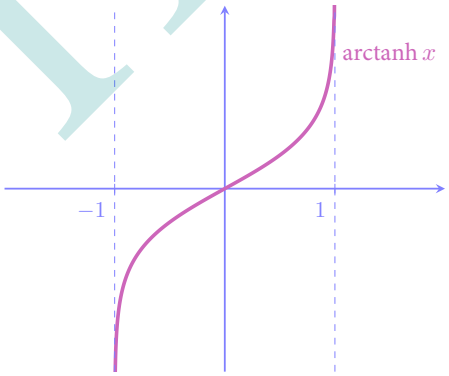
Αντίστροφη Υπερβολικού Ημιτόνου



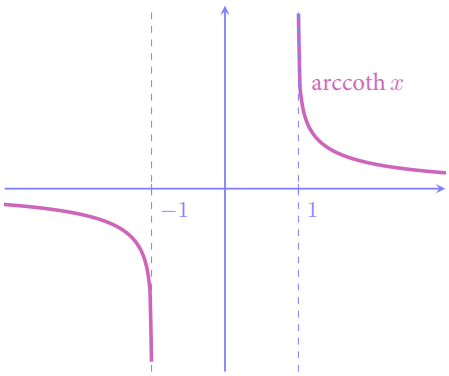
Αντίστροφη Υπερβολικού Συνημιτόνου



Αντίστροφη Υπερβολικής Εφαπτομένης



Αντίστροφη Υπερβολικής Συνεφαπτομένης



ΠΡΟΣΗΜΟ